Name: _____ Solutions _____ PID: _____

Instructions:

1. **DO NOT OPEN THIS EXAM UNTIL YOU ARE INSTRUCTED TO DO SO.**

2. Print your full name and your PID on your exam. Then finish reading these instructions, and **sign the bottom of this page.**

3. Without fully opening the exam, check the page numbers in this exam booklet. Including this cover page, you should have 7 different pages. If you do not, please request another copy immediately.

4. Neither books nor scratch paper are needed for this exam. Clear your desk of everything but this booklet, your pencils and your calculator. If you need more space to write your solutions, use the backs of the exam pages.

5. Calculators are not to be shared. Do not ask your instructor any questions about the use of your calculator. Only those calculators appropriate for MTH 124 (as specified in the course syllabus) are allowed for use during this exam.

6. All electronic equipment (such as cell phones, mp3 players, etc) must be turned off and stored away during the exam time.

7. Crib sheets (pre-compiled lists of formulas or other information) either written or in a calculator are specifically forbidden. **Use of a crib sheet of any kind on this exam will result in an automatic zero grade.**

8. **No talking is allowed during the exam.**

9. The problems on this exam vary in difficulty. You should try to solve these problems in an order that will maximize your score. Solve all the easier problems first, then go back to the ones that require more thought.

10. Unless otherwise indicated, **SHOW ALL YOUR WORK**. If no work is shown, no partial credit can be awarded. Even for calculator solutions, you should include relevant information, such as the equation to be solved, the function whose graph is to be sketched, etc. Your work and answer need to be accurate and relevant to receive points.

11. Unless you are specifically instructed to do otherwise, DO NOT ROUND YOUR ANSWERS – GIVE EXACT ANSWERS.

12. You will be given **exactly 50 minutes** for this exam.

13. *Any student not following the above instructions nor behaving according to the above instructions during the exam may have their exam confiscated and points deducted.*

I have read and fully understood all of the above instructions: **Signature:** _____ *Solutions* _____

Problem	1	2	3	4	5	6	7	TOTAL
Points								
Maximum	30	10	12	15	14	5	14	100

1. The following question has five parts. Clearly state what rules or properties of derivatives you are using. $[6+6+6+6+6 = 30]$

   (a) Find the derivative of $f(x) = 4 - \sqrt[3]{x^2} - x^{-2}$.

   (Power rule)

   $$\frac{d}{dx}\left[4 - \sqrt[3]{x^2} - x^{-2}\right]$$

   $$= \frac{d}{dx}4 - \frac{d}{dx}\sqrt[3]{x^2} - \frac{d}{dx}x^{-2}$$

   $$= 0 - \frac{d}{dx}x^{2/3} - \frac{d}{dx}x^{-2}$$

   $$\longrightarrow -\frac{2}{3}x^{-1/3} - (-2x^{-3})$$

   $$= \boxed{-\frac{2}{3}x^{-1/3} + 2x^{-3}}$$

   (b) Find $g(1)$, $g'(1)$ and $g''(1)$ given that $g(t) = -2t^3 - 3t^2 + t - 4$.

   Power rule $\Rightarrow$ $g'(t) = -2(3t^2) - 3(2t) + 1$

   $$= -6t^2 - 6t + 1$$

   $$g''(t) = -6(2t) - 6(1)$$

   $$= -12t - 6$$

   $$\boxed{\begin{array}{l} g(1) = -8 \\[4pt] g'(1) = -11 \\[4pt] g''(1) = -18 \end{array}}$$

   (c) Find the derivative of $h(s) = 8^s - 2 \cdot 3^s$.

   Using $\frac{d}{ds}a^s = (\ln a)a^s$:

   $$h'(s) = \boxed{(\ln 8)8^s - (2\ln 3)3^s}$$

   (d) Find the derivative of $f(y) = (3y^2 - 9)^{10}$.

   $f(y) = z^{10}$ where $z = 3y^2 - 9$

   $$\frac{df}{dz} = 10z^9, \qquad \frac{dz}{dy} = 6y$$

   Chain rule

   $$\Rightarrow \frac{df}{dy} = 10z^9 \cdot 6y = \boxed{10(3y^2-9)^9 \cdot 6y}$$

   $$= \boxed{60y(3y^2-9)^9}$$

   (e) Find the derivative of $g(x) = \ln(1 + 3x)$.

   $g(x) = \ln z$ where $z = 1 + 3x$

   $$\frac{dg}{dz} = \frac{1}{z} = \frac{1}{1+3x} \qquad \frac{dz}{dx} = 3$$

   $$\therefore \boxed{\frac{dg}{dx} = \frac{3}{1+3x}}$$

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2. Consider the function $f(x) = x^2 - 4x + 3$. $[5 + 3 + 2 = 10]$

(a) Find the equation of the tangent line to f at $x = 3$.

$(*)$ $f'(x) = 2x - 4$

$f'(3) = 2 \cdot 3 - 4 = 2$
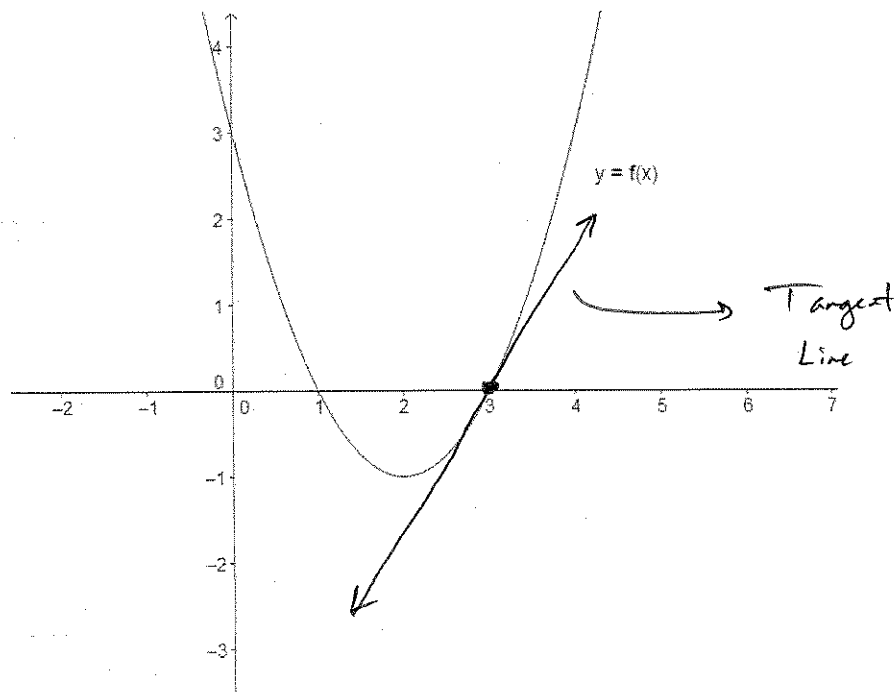
$\Rightarrow \quad y = 2x + b$

$f(3) = 9 - 12 + 3 = 0$

$0 = 2 \cdot 3 + b$

$\Rightarrow \quad b = -6$

$$\boxed{y = 2x - 6}$$

(b) The graph of f is given below. Sketch the tangent line on the same axes.

y = f(x)

Tangent Line

(c) Find $f''(x)$ and explain what this tells you about the graph of f.

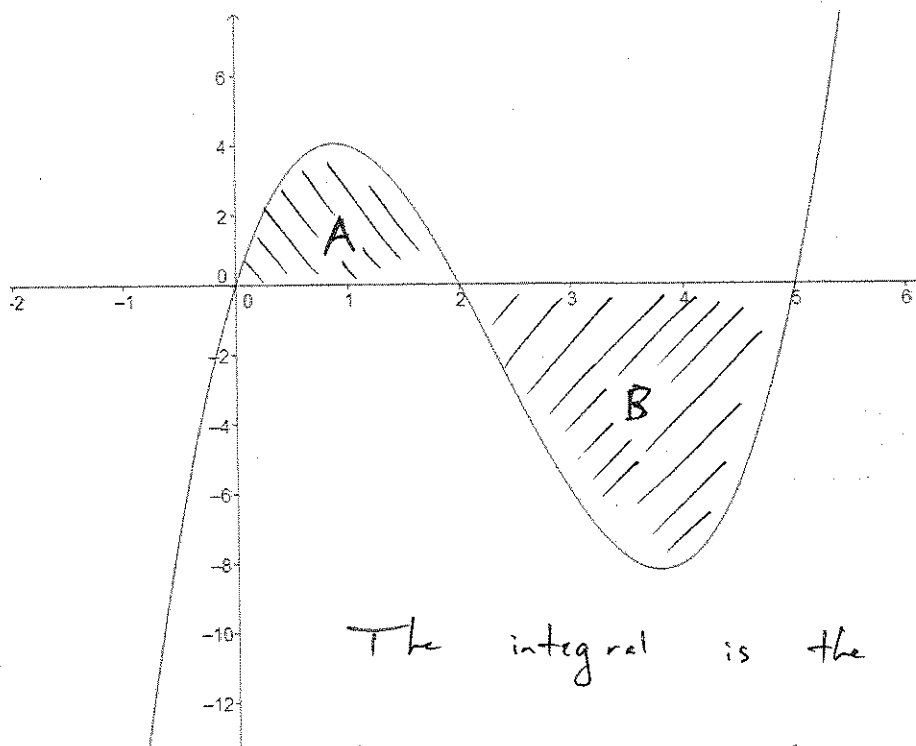From $(*)$: $f'(x) = 2x - 4$

$\Rightarrow \quad f''(x) = 2$

So f'' is always positive $\Rightarrow$ $\boxed{\text{Concave up}}$

3. The graph $f(t) = t(t-2)(t-5)$ is given on the axes below. Explain clearly what

$$\int_0^5 t(t-2)(t-5)dt$$

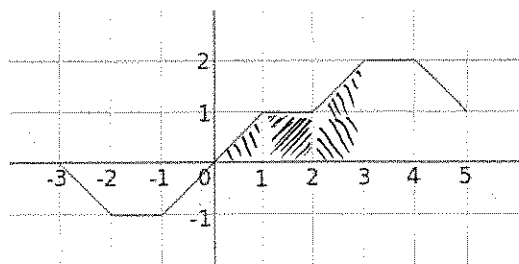means by illustrating areas on the graph. [12]

The integral is the difference in areas. Since the graph of f lies below the x-axis for $2 \le x \le 5$, interpret the area negatively.

So $\boxed{\int_0^5 t(t-2)(t-5)dt = \begin{pmatrix} \text{area of} \\ A \end{pmatrix} - \begin{pmatrix} \text{area of} \\ B \end{pmatrix}}$

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4. The graph of a derivative $f'(x)$ is given below. [15]

Fill in the table of values for $f(x)$ given that $f(0) = 2$.

x	-2	-1	0	1	2	3	4
$f(x)$	3.5	2.5	2	2.5	3.5	5	7

Using the fundamental theorem:

$$f(1) - f(0) = \int_0^1 f'(t)\,dt = \text{area of triangle}$$

$$= \tfrac{1}{2}$$

So $f(1) = \tfrac{1}{2} + f(0) = 2.5$

Likewise, $f(2) - f(1) = \int_1^2 f'(t)\,dt = 1 = \text{area of box}$

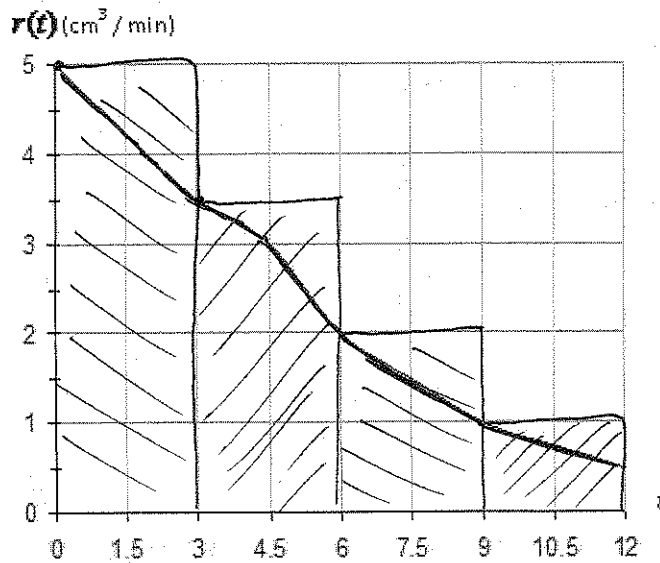$$f(3) - f(2) = 1 + \tfrac{1}{2} = 1.5 = \text{area of}$$

$$f(4) - f(3) = 2$$

interpret area negatively!

Other side: $f(0) - f(-1) = \int_{-1}^0 f'(t)\,dt = -\tfrac{1}{2}$

$\Rightarrow f(-1) = f(0) + \tfrac{1}{2}$.

$f(-1) - f(-2) = \int_{-2}^{-1} f'(t)\,dt = -1 \Rightarrow f(-2) = f(-1) + 1$.

5. The graph below shows the rate $r(t)$ of helium leaking out of a balloon in cubic centimeters per minute as a function of t, in minutes. $[2 + 2 + 10 = 14]$

(a) What are the correct units for $\int_0^9 r(t)dt$?

$$\left(\text{units of } r \right) \left(\text{units of } t \right)$$

$$= \left(\frac{cm^3}{min} \right) \left(min \right) = \boxed{cm^3} \quad \longrightarrow \quad \text{Accumulated rate of change.}$$

(b) In the context of this problem, what is the practical meaning of $\int_3^{12} r(t)dt$?

The amount of helium that leaks out of the balloon between 3 and 12 minutes.

(c) Using the graph, estimate how much helium was lost between 0 and 12 minutes. Use a left-hand sum with 4 rectangles to estimate.

Areas of rectangles

$$= \left(5 \ \frac{cm^3}{min} \right) \left(3 \ min \right) + \left(3.5 \frac{cm^3}{min} \right) \left(3 min \right)$$

$$+ \left(2 \ \frac{cm^3}{min} \right) \left(3 \ min \right) + \left(1 \ \frac{cm^3}{min} \right) \left(3 \ min \right)$$

$$= \left[15 + 10.5 + 6 + 3 \right] \left(\frac{cm^3}{min} \cdot min \right)$$

$$= \boxed{34.5 \ cm^3}$$

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6. Find $F'(x)$ and determine whether F is increasing or decreasing. [5]

$$F(x) = \int_0^x 5^t dt$$

From Second Fundamental Thm,

$$\boxed{F'(x) = 5^x}$$

Derivative F' is positive $\Rightarrow$ $\boxed{F \text{ increasing}}$

7. The marginal cost function for a company is given by

$$C'(q) = \frac{75}{\ln(q+20)}$$

where q is the quantity produced. $[6 + 4 + 4 = 14]$

(a) If the fixed cost is \$1000, find the total cost to produce 100 items.

Fixed Cost + Variable Cost

$$= 1000 + \int_0^{100} \frac{75}{\ln(q+20)} dq \approx \boxed{\$ 2835.81}$$

(b) If each item is sold for \$30, what is the profit (or loss) on the first 100 items?

Revenue $= (\$ 30 \text{ per item})(100 \text{ items}) = \$3000$

$\Rightarrow$ Profit $= \$3000 - \$2835.81$

$= \boxed{\$164.19}$

(c) Find the marginal profit on the 101st item.

$$M\pi = MR - MC$$

$$= \$30 - \underbrace{C'(100)}_{\displaystyle \frac{75}{\ln(100+20)}}$$

$$\approx \boxed{\$14.33}$$